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93202Q

932022

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2008
Mathematics with Calculus

2.00 pm Monday 17 November 2008
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

Pull out the Formulae and Tables Booklet S–CALCF from the centre of this booklet.

There are FIVE questions in this booklet. Answer ALL questions.

Write your answers in the Answer Booklet 93202A.

Show ALL working.

Start your answer to each question on a NEW page. Number each question and part question carefully.

Check that this booklet has pages 2–6 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

You have three hours to complete this examination.

QUESTION ONE (8 marks)

(a) We are always looking for more efficient ways to store and stack things.
Cross sections tell us a lot about the stack and the space being filled.
Figure 1 shows the cross section of a hexagonal stack.

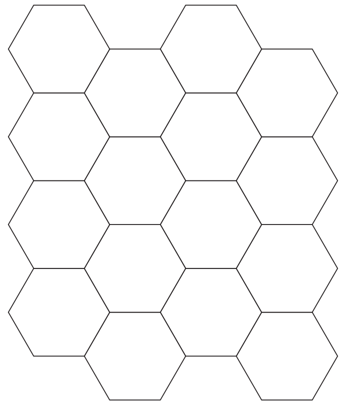

Figure 1

Show that the area of a regular hexagon with edge length s millimetres is $\frac{3\sqrt{3}}{2}s^2$ square millimetres.

Hence show that the total area of the hexagonal stack in Figure 1 is $24\sqrt{3}s^2$ square millimetres.

(b) Although a single cell of a bee's honeycomb has a hexagonal base, it is not a hexagonal prism. The complete cell more commonly has the shape shown in Figure 2.

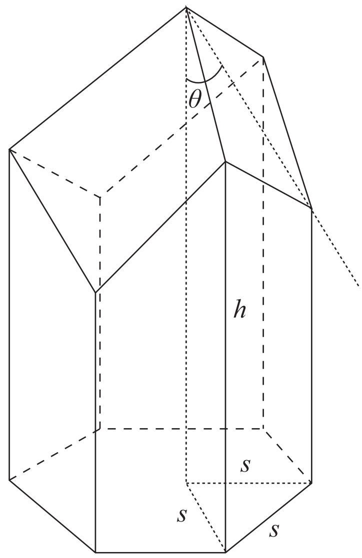

Figure 2

The surface area of this cell is given by

$$A = 6hs + \frac{3}{2}s^2\left(\frac{-\cos\theta}{\sin\theta} + \frac{\sqrt{3}}{\sin\theta}\right)$$

where h, s, θ are as shown in Figure 2.
Keeping h and s fixed, for what angle, θ, is the surface area a minimum?
You do not need to prove it is a minimum.

(c) Another cell, as described in part (b) above, has s not fixed, but increasing at a rate proportional to $\sin\theta$, where $0 < \theta < \frac{\pi}{2}$.

This rate is equal to $\sqrt{2}$ when $\theta = \frac{\pi}{4}$.

Keeping h fixed, at what rate is the surface area decreasing when θ is equal to the value found in part (b) above? Give your answer in surd form, in terms of h and s.

QUESTION TWO (8 marks)

(a) Solve the equation $(z + 1)^3 = 8$, where z is a complex number.
Give your answers in terms of w, where w is a complex number and $w^3 = 1$.
Hence, or otherwise, show that the sum of the roots of $(z + 1)^3 = 8$ is -3.

(b) Solve the equation $(z + 1)^3 = 8(z - 1)^3$, giving exact answers in the form $a + ib$, where $i = \sqrt{-1}$.

(c) Plot the three solutions to the equation in part (b) in the complex plane as the points A, B, and C.

Plot the three solutions to the equation

$$\left(\frac{1}{z}+1\right)^3 = 8\left(\frac{1}{z}-1\right)^3$$

in the complex plane as the points E, F, and G.

Show that the ratio of the areas of the two triangles ABC and EFG is given by:

$$\frac{\Delta ABC}{\Delta EFG} = \frac{81}{49}$$

QUESTION THREE (8 marks)

(a) $\frac{A}{x} + \frac{B}{P-x} = \frac{1}{x(P-x)}$ where x is a variable, and P is a constant.

Find A and B in terms of P.

(b) When a rumour about a teacher is started in a school of size P students, it spreads at a rate (in students per day) that is proportional to the product of the number of students who know the rumour, N, and those who do not.

Find an expression for the number of students, N, who know the rumour after t days.

(c) For a particular rumour about a teacher, 0.5% of the students know the rumour initially. The principal will need to act to stop the rumour once more than half the school's students know it. When $\frac{1}{5}$ of the students know the rumour, the number who know the rumour is increasing at a rate of $0.08P$ students per day.

How long will it be before the principal must act?

QUESTION FOUR (8 marks)

(a) If $f(x)$, $g(x)$, and $h(x)$ are real functions of x, show that

when $h(x) = \left[f(x)\right]^{g(x)}$

then $h'(x) = \left[f(x)\right]^{g(x)} \left(g'(x)\ln[f(x)] + g(x)\dfrac{f'(x)}{f(x)} \right)$

(b) Using the 'Super-Power rule' in part (a) above, or otherwise, find

$$\int (\ln x)^x p'(x)\,\mathrm{d}x$$

when $p(x) = x\ln(\ln x)$.

(c) Find the **condition** for $g(x)$ to be exactly divisible by $x - 1$, for

$g(x) = -(2x - a)^4 + bx + c$

where a, b and c are constants.

Hence or otherwise, fully factorise $g(x)$ and find ALL solutions of the equation

$-(2x - 1)^4 + 8x - 7 = 0.$

QUESTION FIVE (8 marks)

A curve is defined by its parametric equations

$x = \sec\theta - 1$, $y = 2\tan\theta + 2$, where $-\pi < \theta < \pi$, and $\theta \uparrow \pm\frac{\pi}{2}$.

(a) Find the Cartesian equation of the curve.

(b) The normal at the point P, where $\theta = \frac{\pi}{4}$, meets the curve again, at the point Q, in the region where $x < 0$.

Find the y-coordinate of Q in the form $\frac{a}{b}$, where a and b are integers.

(c) The tangent at the point R is parallel to the tangent at P.

Find the y-coordinate of the point S where the normal at R meets the curve again, in the region where $x > 0$.